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LETTER TO THE EDITOR

Factorized representation of Felder's elliptic quantum group

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Abstract. We constructed a factorized representation of the Felder's elliptic quantum group by using the intertwining vectors, and further derived its explicit form which is, in a special case, identical with what Bo-yu Hou *et al* gave, i.e. a cyclic representation of the Sklyanin algebra up to a factor.

The elliptic quantum group [2–4] has recently attracted much attention. In this letter, we will study the factorized representation of Felder's elliptic quantum group in terms of intertwining vectors, originally introduced in [5], which relate the interaction-round-a-face (IRF) model [6, 7] and the Belavin's vertex model [8], i.e. vertex-IRF correspondence [1, 9–11]. Furthemore, we will also derive its explicit form.

We introduce first the Z_n-symmetric Belavin's vertex model and the face model. Let V be a complex n-dimensional vector space with the standard orthonormal basis $\{\epsilon_i\}_{i\in Z_n}$. Both g and h are the linear operators acting on V, and satisfy $g\epsilon_i = \omega^i\epsilon_i$ and $h\epsilon_i = \epsilon_{i-1}$ with $\omega = \exp(2\pi \mathrm{i}/n)$. The state variables of this vertex model defined on a two-dimensional lattice $\mathcal{L}$ take values in Z_n-spin. Its Boltzmann weight for a single vertex give a matrix representation of a linear map $R : V\otimes V \to V\otimes V$ in the basis $\{\epsilon_i\otimes\epsilon_k\}$

$$R(\epsilon_i\otimes\epsilon_k) = \sum_{j,l\in Z_n} \epsilon_l\otimes\epsilon_j \; R_{ik}^{jl}.$$

The Z_n-symmetric Belavin R-matrix [1, 10] can be written as

$$R(z) = P\sum_{(\alpha_1,\alpha_2)\in Z_n^2} \frac{\sigma_\alpha(z+\gamma'/n)}{\sigma_\alpha(\gamma'/n)}(g^{\alpha_1}h^{\alpha_2})\otimes(g^{\alpha_1}h^{\alpha_2})^{-1}$$

where $P(x,y) = (y,x)$ with $(x,y)\in V\otimes V$, $\alpha = (\alpha_1,\alpha_2)\in Z_n\otimes Z_n$, $\tau\in C$ with $\mathrm{Im}\,\tau > 0$ and

$$\sigma_\alpha(z) \equiv \theta\begin{pmatrix}\frac12+\alpha_2/n\\ \frac12+\alpha_1/n\end{pmatrix}(z,\tau) \qquad \theta(z,\tau) = \theta\begin{pmatrix}\frac12\\ \frac12\end{pmatrix}(z,\tau)$$

$$\theta\begin{pmatrix}a\\ b\end{pmatrix}(z,\tau) = \sum_{n\in Z}\exp\{\mathrm{i}\pi(n+a)^2\tau + 2\mathrm{i}\pi(n+a)(z+b)\}$$

$$\theta^{(j)}(z) = \theta\begin{pmatrix}\frac12 - j/n\\ \frac12\end{pmatrix}(z, n\tau) \qquad \text{for } n\geqslant 2.$$

The integrable condition of the system is the Yang–Baxter equation (YBE):

$$R^{12}(z_1-z_2)R^{23}(z_1)R^{12}(z_2)=R^{23}(z_2)R^{12}(z_1)R^{23}(z_1-z_2)$$

where V_i are copies of C^n, and R^{ij} acts only on ith and jth spaces. Note that there is a small difference between some symbols in this letter and those of other papers.

On the other hand, there is a relative interaction-round-a-face (IRF) model on the dual lattice $\mathcal{L}^*$. For convenience, we denote, by $W(a,b,c,d,z)$, its boltzmann weight for a state configuration $\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ round a face with $a,b,c,d\in h^*$ which is the weight space of the Lie algebra sl_n, namely the dual space of the Cartan subalgebra h of sl_n. The only non-vanishing Boltzmann weights [10] can be represented as

$$\begin{aligned} W(m,m-e_j,m-2e_j,m-e_j,z)&=\frac{\theta(z+\gamma')}{\theta(\gamma')}\\ W(m,m-e_j,m-e_j-e_k,m-e_j,z)&=\frac{\theta(z+\gamma' m_{jk})}{\theta(\gamma' m_{jk})}\qquad (j\neq k)\\ W(m,m-e_j,m-e_j-e_k,m-e_k,z)&=\frac{\theta(z)\theta(\gamma' m_{jk}-\gamma')}{\theta(\gamma')\theta(\gamma' m_{jk})}\qquad (j\neq k) \end{aligned} \tag{1}$$

where γ' is a parameter, $m_{jk}=m_j-m_k$, and $e_j=(0,\ldots,1,\ldots,0)$, 1 is at the jth place. They satisfy the YBE:

$$\begin{aligned} &\sum_g W(a,b,c,g,z_1-z_2)W(g,c,d,e,z_1)W(a,g,e,f,z_2)\\ &\qquad=\sum_g W(b,c,d,g,z_2)W(a,b,g,f,z_1)W(f,g,d,e,z_1-z_2). \end{aligned} \tag{2}$$

However, there are the following vertex-IRF correspondence:

$$\sum_{i',j'} R(z_1-z_2)^{ij}_{i'j'}\phi(z_1)^{a\ i'}_b\phi(z_2)^{b\ j'}_c=\sum_d \phi(z_2)^{a\ j}_d\phi(z_1)^{d\ i}_c W(a,b,c,d,z_1-z_2) \tag{3}$$

$$\sum_{i',j'} \bar\phi(z_2)^{a\ j'}_d\bar\phi(z_1)^{d\ i'}_c R(z_1-z_2)^{i'j'}_{ij}=\sum_b W(a,b,c,d,z_1-z_2)\bar\phi(z_1)^{a\ i}_b\bar\phi(z_2)^{b\ j}_c. \tag{4}$$

The above $\phi(z)^b_a$ is the incoming intertwining vector defined as

$$\begin{aligned} &\phi(z)^b_a={}^t(\phi(z)^{b\ 0}_a,\phi(z)^{b\ 1}_a,\ldots,\phi(z)^{b\ n-1}_a)\\ &\phi(z)^{b\ i}_a=\begin{cases}\theta^{(i)}(z+n\gamma' m_j) & \text{if } b-a=e_j\\ 0 & \text{otherwise}\end{cases}\\ &b=(m_0,m_1,\ldots,m_{n-1}). \end{aligned}$$

The above $\bar\phi(z)^b_a$ is the outgoing intertwining vector defined as follows

$$\begin{aligned} &\bar\phi(z)^b_a=(\bar\phi(z)^{b\ 0}_a,\bar\phi(z)^{b\ 1}_a,\ldots,\bar\phi(z)^{b\ n-1}_a)\\ &\bar\phi(z)^{b\ i}_{b-e_j}=B(z)_{ij}/\det A \end{aligned}$$

where A is an $n\times n$ matrix whose elements are A_{ij} with $i,j\in Z_n$ given by

$$A_{ij}=\phi(z)^{b\ i}_{b-e_j}=\theta^{(i)}(z+n\gamma' m_j)$$

and B is the cofactor matrix of A. The elements of B are denoted by B_{ij}.

We are now in the position to introduce the Felder's elliptic quantum group [2, 3] which is closely related to the IRF model. Let $\tilde V$ be an n-dimensional diagonalizable h-modules with the basis e_j defined as before, and $\tilde R(z,\lambda)\in \mathrm{End}(\tilde V\otimes\tilde V)$ the R-matrix of the elliptic quantum group $E = E_{\tau,\gamma}(sl_n)$ with $z \in C$ and an addition variable $\lambda = (\lambda_1, \ldots, \lambda_n) \in h^*$. It is a solution of the dynamical YBE

$$\begin{aligned}&\tilde{R}(z_1 - z_2, \lambda - \gamma h^{(3)})^{12} \tilde{R}(z_1, \lambda)^{(13)} \tilde{R}(z_2, \lambda - \gamma h^{(1)})^{23} \\ &\quad = \tilde{R}(z_2, \lambda)^{23} \tilde{R}(z_1, \lambda - \gamma h^{(2)})^{13} \tilde{R}(z_1 - z_2, \lambda)^{12}. \end{aligned} \tag{5}$$

where $\tilde{R}(z, \lambda - \gamma h^{(3)})^{12}$ acts on a tensor $v_1 \otimes v_2 \otimes v_3$ as $\tilde{R}(z, \lambda - \gamma\mu_3) \otimes \mathrm{Id}$ if v_3 has weight μ_3. Let $E_{i,j}$ be the $n \times n$ matrix satisfying $E_{i,j} e_k = \delta_{j,k} e_i$, then the formula for $\tilde{R}$ is

$$\tilde{R}(z, \lambda) = \sum_{i=1}^{n} E_{i,i} \otimes E_{i,i} + \sum_{i \neq j} \alpha(z, \lambda_{ij}) E_{i,i} \otimes E_{j,j} + \sum_{i \neq j} \beta(z, \lambda_{ij}) E_{i,j} \otimes E_{j,i} \tag{6}$$

where $\lambda_{ij} = \lambda_i - \lambda_j$ and

$$\alpha(z, \lambda) = \frac{\theta(z)\theta(\lambda + \gamma)}{\theta(z - \gamma)\theta(\lambda)} \qquad \beta(z, \lambda) = -\frac{\theta(z + \lambda)\theta(\gamma)}{\theta(z - \gamma)\theta(\lambda)}.$$

The elliptic quantum group $E = E_{\tau,\gamma}(sl_n)$ is an algebra generated by a meromorphic function of a variable h and the matrix elements of a matrix $L(z, \lambda) \in \mathrm{End}(\tilde{V})$ with non-commutative entries, subject to the following equation

$$\begin{aligned}&\tilde{R}(z_1 - z_2, \lambda - \gamma h^{(3)})^{12} L(z_1, \lambda)^{(13)} L(z_2, \lambda - \gamma h^{(1)})^{23} \\ &\quad = L(z_2, \lambda)^{23} L(z_1, \lambda - \gamma h^{(2)})^{13} \tilde{R}(z_1 - z_2, \lambda)^{12}. \end{aligned} \tag{7}$$

On the other hand, there is an important relation between the $\tilde{R}$-operator and the Boltzmann weight $W(a, b, c, d, z)$ of the IRF model:

$$\frac{\theta(z - \gamma)}{-\theta(\gamma)} \tilde{R}(z, \gamma m)\ e_j \otimes e_k = \sum_{i=j,k} W(m, m - e_j, m - e_j - e_k, m - e_i, z) e_i \otimes e_l$$

where $\gamma = -\gamma'$, $e_i = e_j + e_k - e_l$. Thus, we can write down the component form of the equation (7) in terms of the Boltzmann weight,

$$\begin{aligned}&\sum_{j=l,l'} W(a, b, c, d, z_1 - z_2) L^{(1)}(z_1, \lambda)_k^j L^{(2)}(z_2, \lambda - \gamma h^{(1)})_{k'}^{j'} \\ &\quad = \sum_{j_1=k,k'} L^{(2)}(z_2, \lambda)_{j_1}^{l'} L^{(1)}(z_1, \lambda - \gamma h^{(2)})_{i_1}^{l} W(a', b', c', d', z_1 - z_2) \end{aligned} \tag{8}$$

where

$$\gamma a = \lambda - \gamma h^{(3)} = \lambda - \gamma\mu_3 \qquad \gamma a' = \lambda \qquad \gamma = -\gamma' \tag{9}$$

$$a - b = e_j \qquad b - c = e_{j'} \qquad a - d = e_{l'} \qquad d - c = e_l \tag{10}$$

$$a' - b' = e_k \qquad b' - c' = e_{k'} \qquad a' - d' = e_{j_1} \qquad d' - c' = e_{i_1} \tag{11}$$

$$e_j + e_{j'} = e_l + e_{l'} \qquad e_k + e_{k'} = e_{i_1} + e_{j_1}. \tag{12}$$

At present, we will derive the factorized representation by using vertex-IRF correspondence.

Multiply both sides of equation (4) from the right by $\phi(z_1 + \xi)_{b'}^{a'\ i} \phi(z_2 + \xi)_{c'}^{b'\ j}$, and sum over i, j, then the right-hand side (RHS) of the equation becomes

$$\begin{aligned}\mathrm{RHS} &= \sum_b \sum_{i,j} W(a, b, c, d, z_1 - z_2) \bar{\phi}(z_1)_b^{a\ i} \bar{\phi}(z_2)_c^{b\ j} \phi(z_1 + \xi)_{b'}^{a'\ i} \phi(z_2 + \xi)_{c'}^{b'\ j} \\ &= \sum_b W(a, b, c, d, z_1 - z_2) \left[\sum_i \bar{\phi}(z_1)_b^{a\ i} \phi(z_1 + \xi)_{b'}^{a'\ i} \right]\end{aligned}$$

$$
\begin{aligned}
&\times\left[\sum_j \bar\phi(z_2)_c^{b\ j}\phi(z_2+\xi)_{c'}^{b'\ j}\right]\\
&= \sum_b W(a,b,c,d,z_1-z_2)\tilde L(z_1)_{bb'}^{aa'}\tilde L(z_2)_{cc'}^{bb'}
\end{aligned}
$$

where $\tilde L(z_1)_{bb'}^{aa'} = \sum_i \bar\phi(z_1)_b^{a\ i}\phi(z_1+\xi)_{b'}^{a'i}$. The left-hand side (LHS) of the equation becomes

$$
\begin{aligned}
\text{LHS} = &\sum_{i,j,i',j'} \bar\phi(z_2)_d^{a\ j'}\bar\phi(z_1)_c^{d\ i'} R(z_1-z_2)_{ij}^{i'j'}\phi(z_1+\xi)_{b'}^{a'\ i}\phi(z_2+\xi)_{c'}^{b'\ j}\\
&\text{from equation (3)}\\
&= \sum_{i',j'} \bar\phi(z_2)_d^{a\ j'}\bar\phi(z_1)_c^{d\ i'}\left[\sum_{d'}\phi(z_2+\xi)_{d'}^{a'\ j'}\phi(z_1+\xi)_{c'}^{d'\ i'}\right.\\
&\left.\times W(a',b',c',d',z_1-z_2)\right]\\
&= \sum_{d'}\sum_{j'i'}[\bar\phi(z_2)_d^{a\ j'}\phi(z_2+\xi)_{d'}^{a'\ j'}][\bar\phi(z_1)_c^{d\ i'}\phi(z_1+\xi)_{c'}^{d'\ i'}]\\
&\times W(a',b',c',d',z_1-z_2)\\
&= \sum_{d'}\tilde L(z_2)_{dd'}^{aa'}\tilde L(z_1)_{cc'}^{dd'}W(a',b',c',d',z_1-z_2).
\end{aligned}
$$

Thus we obtain an equation as follows from RHS = LHS

$$
\sum_b W(a,b,c,d,z_1-z_2)\tilde L(z_1)_{bb'}^{aa'}\tilde L(z_2)_{cc'}^{bb'} = \sum_{d'}\tilde L(z_2)_{dd'}^{aa'}\tilde L(z_1)_{cc'}^{dd'}W(a',b',c',d',z_1-z_2). \tag{13}
$$

If $L^1(z_1,\lambda)_k^{j'} = \tilde L(z_1)_{bb'}^{aa'} = \sum_i \bar\phi(z_1)_b^{a\ i}\phi(z_1+\xi)_{b'}^{a'i}$ with equations (9)–(12) satified, then equation (13) states that $L^{(1)}(z_1,\lambda)_k^j$ gives the factorized representation of the elliptic quantum group defined in equation (7) or (8). Furthermore, we will derive a more explicit form as follows.

$$
\begin{aligned}
L^{(1)}(z,\lambda)_i^{i'} &= \sum_i \bar\phi(z)_b^{a\ i}\phi(z+\xi)_{b'}^{a'i} \qquad a-b=e_j \qquad a'-b'=e_k\\
&= \sum_i \bar\phi(z)_{a-e_j}^{a\ i}\phi(z+\xi)_{a'-e_k}^{a'\ i}\\
&= \sum_i \frac{B_{ij}(z)}{\det A(z)}\phi(z+\xi)_{b'-e_k}^{b'\ i}\\
&= \frac{1}{\det A(z)}\sum_i B_{ij}(z)\phi(z+\xi)_{a'-e_k}^{a'\ i}\\
&= \frac{1}{\det A(z)}\begin{vmatrix}
\theta^{(0)}(nz_0) & \cdots & \phi_{a'-e_k}^{a'\ 0}(z+\xi) & \cdots & \theta^{(0)}(nz_{n-1})\\
\theta^{(1)}(nz_0) & \cdots & \phi_{a'-e_k}^{a'\ 1}(z+\xi) & \cdots & \theta^{(1)}(nz_{n-1})\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\theta^{(0)}(nz_{n-1}) & \cdots & \underset{j\text{th column}}{\phi_{a'-e_k}^{a'\ n-1}(z+\xi)} & \cdots & \theta^{(n-1)}(nz_{n-1})
\end{vmatrix}\\
&= \frac{\operatorname{Det} F(z)}{\det A(z)}
\end{aligned} \tag{14}
$$

where

$$\mathrm{Det}\,F(z) = \begin{vmatrix} \theta^{(0)}(nz_0) & \cdots & \theta^{(0)}(nz'_k) & \cdots & \theta^{(0)}(nz_{n-1}) \\ \theta^{(1)}(nz_0) & \cdots & \theta^{(1)}(nz'_k) & \cdots & \theta^{(1)}(nz_{n-1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \theta^{(n-1)}(nz_0) & \cdots & \theta^{(n-1)}(nz'_k) & \cdots & \theta^{(n-1)}(nz_{n-1}) \\ & & j\text{th column} & & \end{vmatrix} \tag{15}$$

$$nz_i = z + n\gamma' a_i \qquad nz'_k = z + \xi + n\gamma' a_i \qquad \gamma = -\gamma' \tag{16}$$

$$a = \gamma^{-1}(\lambda - \gamma h^3) = \gamma^{-1}(\lambda - \gamma\mu_3) \qquad a' = \gamma^{-1}\lambda. \tag{17}$$

From the elliptic function theory, it can be shown [10] that

$$\mathrm{Det}\,A(z) = C_1\sigma_0\left(\sum_{j=0}^{n-1} z_j - \frac{n-1}{2}\right)\prod_{i<i'}\sigma_0(z_i - z_{i'}) \tag{18}$$

$$\mathrm{Det}\,F(z) = C_2\sigma_0\left(\sum_{j'=0}^{n-1} z_{j'} - \frac{n-1}{2} - z_j + z'_k\right)\prod_{j\neq i<j'\neq j}\sigma_0(z_i - z_{j'}) \times \prod_{i<j}\sigma_0(z_i - z'_k)\prod_{i>j}\sigma_0(z'_k - z_i) \qquad \text{for fixed } j,k \tag{19}$$

where both C_1 and C_2 are z_i, z'_k independent constants. Thus, without loss of generality, we will drop them below. From equations (14), (18) and (19), and for fixed j, k, we can obtain the following result:

$$\begin{aligned} L(z,\lambda)^j_k &= \frac{\mathrm{Det}\,F}{\mathrm{Det}\,A} \\ &= \frac{\sigma_0(\sum_{j'=0}^{n-1} z_{j'} - z_j + z'_k - \frac{n-1}{2})}{\sigma_0(\sum_{j'=0}^{n-1} z_{j'} - \frac{n-1}{2})}\frac{\prod_{i<j}\sigma_0(z_i - z'_k)}{\prod_{i<j}\sigma_0(z_i - z_j)}\frac{\prod_{i<j}\sigma_0(z_i - z'_k)}{\prod_{i>j}\sigma_0(z'_k - z_i)} \\ &= \frac{\sigma_0(\sum_{i=0}^{n-1} z_i - z_j + z'_k - \frac{n-1}{2})}{\sigma_0(\sum_{i=0}^{n-1} z_i - \frac{n-1}{2})}\frac{\prod_{i\neq j}\sigma_0(z'_k - z_i)}{\prod_{i\neq j}\sigma_0(z_j - z_i)}. \end{aligned} \tag{20}$$

Substituting equation (16) and (17) into the above equation (20), we obtain the explicit form of factorized representation:

$$L(z,\lambda)^j_k = \frac{\sigma_0(z + \frac{\xi}{n} - \gamma\delta - \gamma a_{kj} - \frac{n-1}{2})}{\sigma_0(z - \gamma\delta - \frac{n-1}{2})}\prod_{i\neq j}\frac{\sigma_0(-\frac{\xi}{n} + \gamma a_{ki})}{\sigma_0(\gamma a_{ji})} \tag{21}$$

where $\delta = \sum_{i=0}^{n-1} a_i$ and $a = \gamma^{-1}(\lambda - \gamma h^{(3)}) = \gamma^{-1}(\lambda - \gamma\mu_3)$.

In summary, we constructed a factorized representation of Felder's elliptic quantum group by intertwining the vectors, and calculated further its explicit form which is, in a special case $j = k$, identical to what Bo-yu Hou *et al* [1] gave, i.e. a cyclic representation of the Sklyanin algebra up to a factor.

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